

ENDOMORPHIC PRESENTATIONS OF BRANCH GROUPS

LAURENT BARTHOLDI

ABSTRACT. We introduce “endomorphisms presentations”, or L -presentations: group presentations whose relations are iterated under a set of substitutions on the generating set, and show that a broad class of groups acting on rooted trees admit explicitly constructible finite L -presentations, generalising results by Igor Lysionok and Said Sidki.

1. INTRODUCTION

In the early 80’s, Rostislav Grigorchuk defined a group, G , endowed with many interesting properties: it is a finitely generated, infinite, torsion group; it has intermediate growth; it has a solvable word problem; it has finite width; etc. There are connections of G to innumerable many branches of mathematics: random walks on graphs, Hecke operators, classification of finite-rank Lie algebras, cryptography, etc.

Already in his early papers [Gri84], Rostislav Grigorchuk showed that G is not finitely presentable. However, Igor Lysionok obtained in [Lys85] a recursively defined, infinite set of relators for G , obtained by iterating a simple letter substitution on a finite set of relators (see Theorem 4.5):

The Grigorchuk group G admits the following presentation:

$$G = \langle a, c, d \mid \sigma^i(a^2), \sigma^i(ad)^4, \sigma^i(adacac)^4 \ (i \geq 0) \rangle,$$

where $\sigma : \{a, c, d\}^* \rightarrow \{a, c, d\}^*$ is defined by $\sigma(a) = aca, \sigma(c) = cd, \sigma(d) = c$.

Rostislav Grigorchuk then used this result to construct a finitely presented amenable but not elementary-amenable group [Gri98], thus answering negatively to an old question by Mahlon Day [Day57]: “can every amenable finitely presented group be obtained from finite groups using exact sequences and unions?” He also proved the independence of the relators, by explicitly computing the Schur multiplier $H_2(G, \mathbb{Z})$ in [Gri99].

A (friendly) competitor of the Grigorchuk group is the Gupta-Sidki group $\overline{\overline{\Gamma}}$, which is also a finitely generated infinite torsion group of subexponential growth. Both groups share other properties, as well — see for instance [BG00, BG99] where they are studied simultaneously. Said Sidki described in [Sid87] a general method yielding recursive presentations of such groups, and for $\overline{\overline{\Gamma}}$ derived an explicit, if somewhat lengthy, presentation.

Narain Gupta also followed a completely different path in obtaining recursively presented torsion groups — namely, he started by defining a presentation, that is recursive but presents no explicit regularity like the presentations considered here, and then proves that the associated group is infinite, torsion and finitely generated [Gup84].

In this paper, we define a general class of presentations, which we call *endomorphisms* or *L-presentations*. As a first approximation, they are given by a generating set, some initial relations, and word substitution rules that produce more relations.

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We start by deriving some of their properties, and give explicit presentations for $\overline{\Gamma}$ and other contracting branch groups (see Definitions 3.3 and 3.4; the main property of a *branch group* is that it contains a subgroup K containing a copy $K_1 \cong K^d$ for some $d \geq 2$, all inclusions having finite index. It is *contracting* if there is a metric on K contracted, up to an additive constant, by the projections $K_1 \rightarrow 1^i \times K \times 1^{d-i-1}$). Our main result on groups acting on rooted trees is the following (see Theorems 3.1 and 3.2):

Theorem 1.1. *Let G be a finitely generated, contracting, semi-fractal, regular branch group. Then G is finitely L -presented. However, G is not finitely presented.*

The Schur multiplier of G has the form $A \oplus B^\infty$ for finite-rank groups A, B .

Definition 1.2. An L -presentation is an expression of the form

$$L = \langle S \mid Q \mid \Phi \mid R \rangle,$$

where S is an alphabet (i.e. a set of symbols), $Q, R \subset F_S$ are sets of reduced words (where F_S is the free group on S), and Φ is a set of free group homomorphisms $\phi : F_S \rightarrow F_S$.

L is *finite* if S, Q, Φ, R are finite. It is *ascending* if Q is empty.

L gives rise to a group G_L defined as

$$G_L = F_S / \left\langle Q \cup \bigcup_{\phi \in \Phi^*} \phi(R) \right\rangle^\#,$$

where $\langle \cdot \rangle^\#$ denotes normal closure and Φ^* is the monoid generated by Φ , i.e. the closure of $\{1\} \cup \Phi$ under composition.

As is customary, we shall identify the presentation L and the group G_L it defines, and write G for both.

The name “ L -presentation” comes both as a homage to Igor Lysionok who discovered such a presentation for the Grigorchuk group G (see Theorem 4.5) and as a reference to “ L -systems” as defined by Aristid Lindenmayer [Lin73] in the early 70’s (see [RS80]), used to model biological growth phenomena.

1.1. Symmetric groups. The purpose of L -presentations is to encode in homomorphisms $\phi \in \Phi$ some regularity of the presentation. Consider for instance the presentations of finite symmetric groups. It is well-known that the following is a presentation of \mathfrak{S}_n , the symmetric group on n objects (see [Bur97, Moo97] for its first occurrences in literature and [Ser93] for other presentations):

$$\begin{aligned} \mathfrak{S}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid & \sigma_i^2 \text{ whenever } 1 \leq i \leq n-1, \\ & (\sigma_i \sigma_{i+1})^3 \text{ whenever } 1 \leq i \leq n-2, (\sigma_i \sigma_j)^2 \text{ whenever } 1 \leq i < j-1 \leq n-2 \rangle. \end{aligned}$$

A shorter ascending L -presentation with the same generators can be obtained if one lets the symmetric group act on itself by conjugation; there remain only 3 orbits of relators under this action. To the point, consider the set $P = \{(1, \dots, n), (1, 2), (3, \dots, n)\}$ generating \mathfrak{S}_n . For each $p \in P$, let it act as ϕ_p on the free group $F_{\sigma_1, \dots, \sigma_{n-1}}$ in such a way that this action is a lift of the action of \mathfrak{S}_n by conjugation on itself, and such that if $\sigma_i^p = \sigma_j$ then $\phi_p(\sigma_i) = \sigma_j$ — a simple way of selecting such a ϕ_p is to pick for each σ_i a word W over $\{\sigma_1, \dots, \sigma_{n-1}\}$ of minimal length representing σ_i^p , and setting $\phi_p(\sigma_i) = W$, extended by concatenation. We then obtain

$$\mathfrak{S}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \{\phi_p\}_{p \in P} \mid \sigma_1^2, (\sigma_1 \sigma_2)^3, (\sigma_1 \sigma_3)^2 \rangle.$$

Indeed all relations σ_i^2 and $(\sigma_i \sigma_{i+1})^3$ are obtained as $\phi_{(1, \dots, n)}^{i-1}(\sigma_1^2)$ and $\phi_{(1, \dots, n)}^{i-1}((\sigma_1 \sigma_2)^3)$, and all relations $(\sigma_i \sigma_j)^2$ are obtained as $\phi_{(1, \dots, n)}^{i-1} \phi_{(3, \dots, n)}^{j-i-2}((\sigma_1 \sigma_3)^2)$. Conversely, all $\phi(r)$ are relations for $\phi \in \{\phi_p\}^*$ and r a relation, since the ϕ are endomorphisms.

Using the same reasoning, one can obtain an ascending L -presentation of \mathfrak{S}_n with only two relators, if one allows more generators:

$$\mathfrak{S}_n = \langle \sigma_{1,2}, \sigma_{1,3}, \dots, \sigma_{n-1,n} \parallel \phi_{(1,2)}, \phi_{(1,\dots,n)} \mid \sigma_{1,2}^2, \sigma_{1,2}\sigma_{2,3}\sigma_{1,3}\sigma_{2,3} \rangle,$$

where $\sigma_{i,j}$ should be interpreted as the transposition (i, j) , and $\phi_p(\sigma_{i,j}) = \sigma_{ip,jp}$.

(This regularity in the presentation is reflected by the fact that $H_2(\mathfrak{S}_n, \mathbb{Z}) = \mathbb{Z}/2$ is very small — see Subsection 2.3.)

Problem 1.3. *Does there exist a bound A such that all symmetric groups can be defined by an ascending L -presentation $\langle S \parallel \Phi \mid R \rangle$ of total length $|S| + |\Phi| + |R| < A$?*

1.2. Other examples. Another example is given by presentations of the free abelian group \mathbb{Z}^n :

$$\mathbb{Z}^n = \langle x_1, \dots, x_n \mid [x_i, x_j] \forall i, j \in \{1, \dots, n\} \rangle.$$

It can be expressed with fewer relators as

$$\mathbb{Z}^n = \langle x_1, \dots, x_n \parallel \phi_1, \phi_2 \mid [x_1, x_2] \rangle,$$

with $\phi_1 : x_1 \mapsto x_2, x_2 \mapsto x_3, \dots, x_n \mapsto x_1$ and $\phi_2 : x_1 \mapsto x_1, x_2 \mapsto x_3, x_3 \mapsto x_4, \dots, x_n \mapsto x_2$. Of course, the main interest of L -presentations is to encode groups that don't even have a finite presentation: consider for instance the group $\mathfrak{S}_\infty \mathbb{Z}$ of permutations of \mathbb{Z} that act like a translation outside a finite interval. It is generated by $\sigma = (1, 2)$ and $\tau : n \mapsto n + 1$:

$$\begin{aligned} \mathfrak{S}_\infty \mathbb{Z} &= \langle \tau, \sigma \mid \sigma^2, [\sigma, \tau^n]^2 \forall n \geq 2, [\sigma, \tau]^3 \rangle \\ &= \langle \tau, \sigma, \bar{\sigma} \mid \sigma \bar{\sigma} \mid \phi \mid \sigma^2, [\sigma, \tau]^3, [\sigma, \bar{\sigma}^{\tau^2}] \rangle \\ &= \langle \tau, \sigma, \bar{\tau} \mid \tau^{-1} \bar{\tau} \mid \psi \mid \sigma^2, [\sigma, \tau]^3, [\sigma, \sigma^{\tau \bar{\tau}}] \rangle \end{aligned}$$

with $\phi(\bar{\sigma}) = \bar{\sigma}^\tau$ and $\psi(\bar{\tau}) = \tau \bar{\tau}$, both preserving the other generators σ and τ . The extra generators $\bar{\sigma}$ and $\bar{\tau}$ are just convenient copies of the generators.

1.3. Outline. The paper is organized as follows: Section 2 contains group-theoretical results on L -presentations. Section 3 describes the main result of this paper, namely that all finitely generated regular branch groups have a finite L -presentation. Section 4 describe the L -presentations of the 5 “testbed” branch groups introduced in [BG00].

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1.4. Notations. For me, g^h denotes $h^{-1}gh$, and the expression $g^{\sum n_i h_i}$ means $\prod h_i^{-1} g^{n_i} h_i$. The commutator $[g, h]$ is $g^{-1}h^{-1}gh$, and X^* is the monoid generated by X . The normal closure of X in G is written $\langle X \rangle^\#$, the normal subgroup of G *normally generated* by X .

2. GROUP-THEORETICAL PROPERTIES

In this section, we are interested in the following questions: which group-theoretical constructions preserve the property of having a finite L -presentation? which groups admit a finite L -presentation?

We shall say a group is *finitely L -presented* if it admits a finite L -presentation.

Remark 2.1. There are finitely L -presented groups that, for some imposed generating set, do not admit a finite L -presentation. This is in contrast with finitely presented groups, for which admitting a finite presentation is independent of the choice of generators — that property is even invariant under quasi-isometries.

For instance, consider the “lamplighter group” of Theorem 4.1, with its finite L -presentation. This group G does not have a finite L -presentation with generators $\{a, t\}$, as can be seen by a careful study of endomorphisms of \mathbb{F}_2 .

Proposition 2.2. *Let G admit a finite ascending L -presentation, and let S' be a finite generating set of G . Then G admits a finite ascending L -presentation with generators S' .*

Proof. The standard proof that being finitely presented involves Tietze transformations, and extends to L -presentations. One changes S into S' by a finite number of “Tietze moves”, which either replace a generator by a product or quotient of generators, or add or delete a generator s along with the relator s .

For an L -presentation $\langle S \mid \Phi \mid R \rangle$, the operations are as follows: if the move was to replace the generator s by $s' = st$, one replaces all instances of s by $s't^{-1}$ in R and the images of $\phi \in \Phi$, modifying them by $\phi(s') = \phi(st)$.

If the move was addition of a generator s to S and R , one extends all $\phi \in \Phi$ by $\phi(s) = 1$. If the move was deletion of s from S and R , one deletes all instances of s in the images of all $\phi \in \Phi$, and adds $\phi(s)$ to R . \square

2.1. Embeddings. We start by some motivation for the study of L -presentations. Recall Graham Higman’s Embedding Theorem

Theorem 2.3 ([Hig61] or Section IV.7 of [LS70]). *A countable group G can be embedded in a finitely presented group \widehat{G} if and only if it is recursively presented.*

The first proof by Higman was unconstructive; since then, explicit constructions of \widehat{G} were given [Aan73, AC80, OS]. They require, however, a good mastery of Turing- or S -machine programming. I am not aware of an explicit finitely presented group containing \mathbb{Q} . In contrast, a finitely L -presented group containing \mathbb{Q} follows: first,

$$\mathbb{Q} = \langle x_1, x_2, \dots \mid x_n x_{n+1}^{-n-1} \forall n \geq 1 \rangle,$$

where x_n should be interpreted as $1/n! \in \mathbb{Q}$. We embed \mathbb{Q} in the finitely generated group

$$G = \langle x, y \mid x^n y^n (x^{n+1} y^{n+1})^{-n-1} \forall n \geq 1 \rangle = \langle x, y \mid yx(x^{n+1} y^{n+1})^n \forall n \geq 1 \rangle$$

through $x_n \mapsto x^n y^n$; now G embeds in the finitely L -presented group H given by

$$H = \langle x, y, a, b, c, d, e, c', d', e' \mid c^{-1}c', d^{-1}d', e^{-1}e', [\langle d, e \rangle, \langle x, y \rangle] \mid \phi_1, \phi_2, \phi_3, \phi_4 \mid yxbac, (e')^{d'}c' \rangle$$

where

$$\phi_1 : \begin{cases} a \mapsto bac \end{cases} \quad \phi_2 : \begin{cases} a \mapsto 1 \\ b \mapsto d^{-1}ybx d \end{cases} \quad \phi_3 : \begin{cases} a \mapsto 1 \\ b \mapsto yex \end{cases} \quad \phi_4 : \begin{cases} c' \mapsto c'c \\ d' \mapsto d'd \\ e' \mapsto e'e \end{cases},$$

it being understood that unspecified generators map to themselves. Indeed $n-1$ applications of ϕ_1 to $yxbac$ yield yxb^nac^n ; then n applications of ϕ_2 yield $yx((d^{-1}y)^nb(xd)^n)^nc^n$; then ϕ_3 and the commutation relations yield $yx(y^{n+1}x^{n+1})^nd^{-n}e^nd^nc^n$. On the other hand, $\phi_4^{n-1}((d')^{-1}e'd'c')$ yields $d^{-n}e^nd^nc^n$, so $yx(y^{n+1}x^{n+1})^n = 1$ in H , whence $x^ny^n = (x^{n+1}y^{n+1})^{n+1}$ in H . Any other sequence of operations ϕ_i would give a long relation containing non- $\{x, y\}$ symbols, so G embeds in H .

Some finitely L -presented groups embed nicely in finitely presented groups; recall that the HNN extension $\Omega(G, H \xrightarrow{\phi} K)$ is *ascending* if $H = G$.

Theorem 2.4. *Let G be finitely L -presented by an ascending L -presentation. Then a finitely presented group \widehat{G} containing G can be effectively constructed. Moreover, \widehat{G} is an ascending HNN extension of G by a finite number of stable letters.*

In case G is amenable, \widehat{G} is a finitely presented amenable group containing G .

Proof. Let $\langle S \parallel \Phi \mid R \rangle$ be a finite ascending L -presentation of G . Consider the group

$$\widehat{G} = \langle S \cup \Phi \mid R \cup \{s^\phi = \phi(s)\}_{s \in S, \phi \in \Phi} \rangle.$$

It is finitely presented, and the map $G \rightarrow \widehat{G}$ defined by sending $s \in S$ to s is a well-defined injective homomorphism, since the $\phi : S^* \rightarrow S^*$ induce injective homomorphisms of G . \square

As a consequence, we may construct finitely generated subgroups of hyperbolic groups that are not hyperbolic. Recall that a set W satisfies the *small cancellation condition* $C'(\epsilon)$ if for any $u, v \in \overline{W}$ the common prefix of u and v has length strictly less than $\epsilon \min\{|u|, |v|\}$, where \overline{W} is the closure of W under taking inverses and cyclic conjugates.

Corollary 2.5. *Let $\langle S \parallel \phi \mid R \rangle$ be a finite ascending L -presentation of G with $R \neq \emptyset$, and let \widehat{G} be the finitely presented group constructed above. Assume that $R \cup \phi(S)$ satisfies the small cancellation condition $C'(1/6)$.*

Then \widehat{G} is a hyperbolic group containing a non-hyperbolic finitely generated, infinitely presented subgroup G . In particular G is not quasi-convex in \widehat{G} .

Proof. It follows from the hypotheses that $\bigcup_{n \geq 0} \phi^n(R)$ is also $C'(1/6)$. Therefore G is not finitely presented, so cannot be hyperbolic. On the contrary, \widehat{G} is finitely presented and its presentation is $C'(1/6)$, so it is hyperbolic. Finally a quasi-convex subgroup of a hyperbolic group would be hyperbolic, so G cannot be quasi-convex. \square

As a simple application of this corollary, consider the group

$$G = \langle x, y \parallel \phi \mid (xy)^7 \rangle,$$

with $\phi(x) = x^7$ and $\phi(y) = y^7$, embedding in

$$\widehat{G} = \langle x, y, \phi \mid x^{\phi^{-7}}, y^{\phi^{-7}}, (xy)^7 \rangle.$$

Proposition 2.6. *If G is finitely presented, then it is finitely L -presented. There are non-finitely L -presented groups, and there are finitely L -presented, but not finitely presented, groups.*

Proof. The first claim is obvious: finite L -presentations with $R = \Phi = \emptyset$ are precisely finite presentations.

There are only countably many finite L -presentations, but uncountably many finitely-generated groups, so “most” groups are not finitely L -presented.

Finally, Theorem 4.1 shows that the “lamplighter group” described there is finitely L -presented, but not finitely presented. \square

Note, however, that it is not trivial to explicitly point at a non-finitely L -presentable group. A group having a non-recursively-enumerable presentation satisfying some small cancellation condition would be an example. The free group in a variety defined by infinitely many identities (they exist by [Ol’70]) is another one. More examples appear in the course of Lemma 2.9.

Proposition 2.7. *If G, H are finitely L -presented groups, then $G * H$ is finitely L -presented.*

If G is finitely L -presented and H, K are isomorphic finitely generated subgroups of G , then the HNN extension $\Omega(G, H \xrightarrow{\psi} K)$ is finitely L -presented.

Proof. Let $\langle S|Q|\Phi|R \rangle$ be a finite L -presentation of G , and let $\langle T|P|\Psi|U \rangle$ be a finite L -presentation of H . A finite L -presentation of $G * H$ is

$$\langle S \cup T | Q \cup P | \Phi \cup \Psi | R \cup U \rangle,$$

where it is understood that each $\phi \in \Phi$ is extended to a homomorphism $\phi : (S \cup T)^* \rightarrow (S \cup T)^*$ by mapping each $t \in T$ to itself; and similarly for each $\psi \in \Psi$.

Let now H be the subgroup of G generated by $T \subset S^*$. A presentation for the HNN extension of G by $\psi : H \rightarrow K$ is

$$\langle S \cup \{\psi\} | Q \cup \{\psi(t)^{-1}t^\psi\}_{t \in T} | \Phi | R \rangle.$$

□

Proposition 2.8. *If G, H are finitely L -presented groups, then any split extension of G by H is finitely L -presented. If H is finitely presented, then any extension of G by H is finitely L -presented.*

Proof. Let $\langle S|Q|\Phi|R \rangle$ be a finite L -presentation of G ; let $\langle T|P|\Psi|U \rangle$ be a finite L -presentation of H ; let X be an extension of G by H , given as $1 \rightarrow G \rightarrow X \rightarrow H \rightarrow 1$. Let σ be a section of H to X ; in case the extension splits, we suppose that σ is a group homomorphism.

Each relator $p \in P$ lifts through σ to an element $g_p \in G$, so we may define $P' = \{pg_p^{-1} | p \in P\}$, a set of relators in X . Since G is normal in X , we also have $s^{\sigma(t)} = g_{s,t} \in G$ for each $s \in S, t \in T$. Consider now the presentation

$$(1) \quad \langle S \cup T | Q \cup P' \cup \{s^t g_{s,t}^{-1}\}_{s \in S, t \in T} | \Phi \cup \Psi | R \cup U \rangle,$$

where it is understood that each $\phi \in \Phi$ is extended to a homomorphism $\phi : (S \cup T)^* \rightarrow (S \cup T)^*$ by mapping each $t \in T$ to itself; and similarly for each $\psi \in \Psi$.

If X is a split extension, then $g_p = 1$ for all $p \in P$, and similarly all $\phi(u)$ (with $u \in U$ and $\phi \in \Phi^*$) are relations in X . If H is finitely presented, we may suppose $U = \emptyset$ and again all relations given in (1) are satisfied.

We have shown that in the cases considered X is a quotient of (1). Let now w be a word in $S \cup T$ equal to 1 in X . The relations $s^t = g_{s,t}$ allow w to be written as $s_1 \dots s_n t_1 \dots t_m$; then projecting to H gives $t_1 \dots t_m = 1$ by applying relations in H . The same relations in (1) will reduce $s_1 \dots s_n t_1 \dots t_m$ to a word in S^* ; the corresponding element of G can be reduced to 1 using relations in G , and these same relations exist in X , so $w = 1 \in X$ and (1) is a presentation of X . □

Note that there are extensions of finitely L -presented groups that are not finitely L -presented; more precisely,

Lemma 2.9. *There are uncountably many non-isomorphic extensions of $\mathbb{Z}/2$ by $\mathbb{Z}/2 \wr \mathbb{Z}$.*

As a consequence, there are uncountably many such extensions that are not finitely L -presented.

Proof. $H^2(\mathbb{Z}/2 \wr \mathbb{Z}, \mathbb{Z}/2) = (\mathbb{Z}/2)^\infty$ — see Subsection 2.3. □

Proposition 2.10. *If G is an finitely L -presented group, then any finite-index subgroup of G is finitely L -presented.*

If $N \triangleleft G$ is finitely generated as a normal subgroup of a finitely L -presented group G , then G/N is finitely L -presented.

Proof. Let $\langle S|Q|\Phi|R \rangle$ be a finite L -presentation of G , and let X be a right transversal of the finite-index subgroup H of G . In view of Proposition 2.8, we may suppose H is normal in G , since any finite-index subgroup is a finite extension of its core, which is normal of finite index.

We then have $G = \bigcup_{x \in X} Hx = \bigcup_{x \in X} xH$. For $g \in G$, let $\bar{g} \in X$ denote its coset representative. By the Reidemeister-Schreier method, H is generated by the finite set $T = \{s^x\}_{x \in X, s \in S}$, and a presentation of H is given by

$$\langle T \mid \{q^x\}_{q \in Q, x \in X} \cup \{\widetilde{\phi(r)^x}\}_{\phi \in \Phi^*, r \in R, x \in X} \rangle,$$

where \widetilde{w} is a rewriting of w as a word over T . Now each $\phi \in \Phi$ induces naturally a monoid homomorphism $\widetilde{\phi}$ over T^* , and since $\widetilde{\phi(r)^x} = \widetilde{\phi}(\widetilde{r^x})$, a finite L -presentation for H is given by

$$\langle T \mid \{\widetilde{q^x}\}_{q \in Q, x \in X} \mid \{\widetilde{\phi}\}_{\phi \in \Phi} \mid \{\widetilde{r^x}\}_{r \in R, x \in X} \rangle.$$

For the second statement of the proposition, let $\langle S \mid Q \mid \Phi \mid R \rangle$ be a finite L -presentation of G and let T be a finite generating set for N . Then

$$\langle S \mid Q \cup N \mid \Phi \mid R \rangle$$

is a finite L -presentation of G/N . □

Proposition 2.11. *If G, H are finitely L -presented groups, and either G is abelian or H is finite, then the restricted wreath product $G \wr H$ is finitely L -presented.*

Problem 2.12. *The corresponding assertion with “finitely L -presented” replaced by “finitely presented” does not hold. Under which conditions does the statement hold, for non-abelian G and infinite H ?*

Proof. If H is finite, then $G \wr H$ is finitely L -presented by Proposition 2.8. Let us assume then that G is abelian. Let $\langle S \mid Q \mid \Phi \mid R \rangle$ be a finite L -presentation of G , and let $\langle T \mid P \mid \Psi \mid U \rangle$ be a finite L -presentation of H . An L -presentation of $G \wr H$ is

$$\langle S \cup T \mid Q \cup P \cup \{[s_1, s_2^h]\}_{s_1, s_2 \in S, h \in H} \mid \Phi \cup \Psi \mid R \cup U \rangle,$$

where it is understood that each $\phi \in \Phi$ is extended to a homomorphism $\phi : (S \cup T)^* \rightarrow (S \cup T)^*$ by mapping each $t \in T$ to itself; and similarly for each $\psi \in \Psi$. This L -presentation is in general not finite, but this can be remedied by introducing new generators \bar{S} in bijection with S and new homomorphisms Ω_T in bijection with T :

$$G \wr H = \langle S \cup T \cup \bar{S} \mid Q \cup P \cup \{s^{-1}\bar{s}\}_{s \in S} \mid \Phi \cup \Psi \cup \Omega_T \mid R \cup U \cup \{[s_1, \bar{s}_2]\}_{s_1, s_2 \in S} \rangle,$$

where $\omega_t \in \Omega_T : (S \cup T \cup \bar{S})^* \rightarrow (S \cup T \cup \bar{S})^*$ is defined by $\omega_t(\bar{s}) = \bar{s}^t$ and $\omega_t(s) = s$ and $\omega_t(t') = t'$. Indeed the new generators \bar{s} do not enlarge G , since $\bar{s} = s$ is a relation; also,

$$[s_1, s_2^h] = [s_1, \bar{s}_2^h] = [s_1, \bar{s}_2^{t_1 \dots t_n}] = [s_1, \omega_{t_1} \dots \omega_{t_n}(\bar{s}_2)] = \omega_{t_1} \dots \omega_{t_n}([s_1, \bar{s}_2]) = 1$$

is a relation, for all $h = t_1 \dots t_n \in H$. □

Problem 2.13. *Let G be a finitely L -presented group generated by S , let H a finitely generated subgroup, and let X be a transversal of H in G which is a regular subset of S^* . Under which extra conditions is H finitely L -presented?*

2.2. Identities. We now show that groups defined by identities are all finitely L -presented. Recall that an identity is a word $w \in F_Y$, and that the group G satisfies the identity w if $f(w) = 1$ for all $f : F_Y \rightarrow G$. For instance, all abelian groups satisfy the identity $[y_1, y_2]$. The free group on S with respect to w is $F_S / \langle f(w) \mid \forall f : F_Y \rightarrow F_S \rangle$. It is the largest group satisfying w , in the sense that every group generated by S and satisfying w is a quotient of it. These groups are also sometimes referred to as *relatively free groups* of finitely based varieties [Neu67]. In that spirit, a group has presentation $\langle S \mid R \rangle$ within a variety if it is the quotient of the free group on S in that variety by the normal closure of R .

Proposition 2.14. *Let G be finitely generated and finitely presented with respect to the identity w . Then G is finitely L -presented.*

Proof. It suffices to prove the claim for a relatively free group, since the quotient of a finitely L -presented group by a finitely normally generated normal subgroup remains finitely L -presented.

Let us then suppose G relatively free, and generated by X , and write $w = w(y_1, \dots, y_n) \in F_Y$. For $x \in X^{\pm 1}$ and $y \in Y$, define the endomorphism ϕ_{xy} of $F_{X \sqcup Y}$ by $\phi_{xy}(y) = xy$, and $\phi_{xy}(s) = s$ for all other $s \in X \sqcup Y$. Then the following is a finite L -presentation of G :

$$\langle X \sqcup Y \mid Y \mid \{\phi_{xy}\}_{x \in X^{\pm 1}, y \in Y} \mid \{w\} \rangle.$$

Indeed write $\Phi = \{\phi_{xy}\}_{x \in X^{\pm 1}, y \in Y}$. Then

$$\begin{aligned} \langle X \sqcup Y \mid Y \mid \Phi \mid \{w\} \rangle &= \langle X \sqcup Y \mid Y \cup \Phi(w) \rangle = \langle X \sqcup Y \mid Y \cup \{w(w_1(X)y_1, \dots, w_n(X)y_n)\} \rangle \\ &= \langle X \mid \{w(w_1(X), \dots, w_n(X))\} \rangle = \langle X \mid f(w) \quad \forall f : F_Y \rightarrow F_X \rangle, \end{aligned}$$

where the $w_i(X)$ are arbitrary words over X , and $f : F_Y \rightarrow F_X$ is given by $f(y_i) = w_i(X)$. \square

As a consequence, the free Burnside groups (defined by the identity $a^n \in F_a$), the rank- r free solvable groups, etc. are finitely L -presented. Moreover:

Corollary 2.15. *Any finitely generated group in the variety \mathfrak{AN}_k of abelian-by-(nilpotent of degree k) groups is finitely L -presented.*

Proof. By [Hal54], every group in the variety \mathfrak{AN}_k is the quotient of the free group in that variety (defined by the identity $[[x_1, \dots, x_k], [y_1, \dots, y_k]] \in F_{x_i, y_i}$) by a finite number of relations. \square

2.3. Schur multipliers. It is well known, by Issai Schur and Heinz Hopf's formula [Bro94, Theorem 5.3], that the Schur multiplier $H_2(G, \mathbb{Z})$ ($= H^2(G, \mathbb{C}^\times)$ for finite groups) of a group G can be computed from a presentation of G ; namely, given a presentation $G = \langle S \mid T \rangle$, we have

$$H_2(G, \mathbb{Z}) = \frac{\langle T \rangle^\# \cap [F_S, F_S]}{[\langle T \rangle^\#, F_S]}.$$

As a consequence, a finitely presented group necessarily has a finite-rank Schur multiplier. (Note, however, that the converse is not true — see Theorem 4.2.) We note that Hopf's formula extends to L -presentations.

Let us first recall a few facts on Schur multipliers; see [Kar87] for further details:

- Norman Blackburn's result [Bla72]

$$(2) \quad H_2(H \wr G, \mathbb{Z}) = H_2(G, \mathbb{Z}) \oplus H_2(H, \mathbb{Z}) \oplus \frac{\{f : G \rightarrow H/H' \otimes H/H'\}}{\{f(x^{-1}) = \tau f(x)\}},$$

where $\tau : H/H' \otimes H/H' \rightarrow H/H' \otimes H/H'$ sends $h \otimes h'$ to $h' \otimes h$, and the f above are just set maps.

- A special case of the Künneth's formula,

$$(3) \quad H_2(G \times H, \mathbb{Z}) = H_2(G) \oplus H_2(H) \oplus (G/G' \otimes H/H'),$$

- Shapiro's lemma: for an exact sequence $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$,

$$(4) \quad H_2(N, \mathbb{Z}) = H_2(G, \mathbb{Z}Q),$$

with the G -action on $\mathbb{Z}Q$ induced by the quotient map $G \rightarrow Q$.

Theorem 2.16. *Let G admit a finite L -presentation $\langle S \mid Q \mid \Phi \mid R \rangle$. Then $H_2(G, \mathbb{Z}) = A \oplus \bigoplus_{\Phi^*} B$, where A and B are finitely-generated abelian groups.*

Proof. Write $F = F_S$, and $W = \langle \Phi^*(R) \rangle^\#$. Consider the group $W/[W, F]$. It is abelian and generated by $\Phi^*(R)$. The maps $\phi \in \Phi$ are such that $\text{coker } \phi$ is finitely generated (by R), and we may filter R along Φ^* . For each $\phi \in \Phi$, write

$$1 \longrightarrow \ker \phi \longrightarrow W \xrightarrow{\phi} W \longrightarrow A_\phi \oplus B_\phi \longrightarrow 1,$$

where A_ϕ splits back into W and B_ϕ does not. Then W lies inside $\bigoplus_{\phi \in \Phi} B_\phi \oplus \bigoplus_{\phi \in \Phi^*} A_\phi$, so W is of the required form. The Schur multiplier is obtained from W by extending by $\langle S \rangle^\#/[F, S]$ (which has finite rank), and restricting to $[F, F]$, both operations preserving the claimed form of $H_2(G, \mathbb{Z})$. \square

It follows, for instance, that $H_2(G, \mathbb{Z})$ may be neither \mathbb{Q} nor $\mathbb{Z}[\frac{1}{n}]$, for a finitely L -presented group. However, the Schur multiplier may be trivial, as in Theorem 4.2, or of infinite-rank, as in the examples of branch groups of Subsection 3.2.

3. BRANCH GROUPS

The purpose of this section is to prove the following general results:

Theorem 3.1. *Let G be a finitely generated, contracting, semi-fractal, regular branch group. Then G is finitely L -presented.*

Theorem 3.2. *Let G be a finitely generated, contracting, semi-fractal, regular branch group. Then G is not finitely presented.*

Even though I am sure that the contracting hypothesis is not needed in Theorem 3.2, I have been unable to prove it without that extra condition.

We start by recalling some definitions from [BG00, Gri00] concerning branch groups. We fix an integer $d \geq 2$, and the finite alphabet $\Sigma = \mathbb{Z}/d\mathbb{Z}$, written $\{1, \dots, d\}$ for convenience. The d -regular rooted tree is the free monoid Σ^* . A tree automorphism $g \in \text{Aut } \Sigma^*$ is a bijective map $g : \Sigma^* \rightarrow \Sigma^*$ that preserves prefixes, i.e. such that $g(\sigma\tau) \in g(\sigma)\Sigma^{|\tau|}$ for all $\sigma, \tau \in \Sigma^*$. There is an isomorphism between the subtree $\sigma\Sigma^*$ and Σ^* , given by left-cancellation of σ . It induces an isomorphism $\pi_\sigma : \text{Aut}(\sigma\Sigma^*) \rightarrow \text{Aut}(\Sigma^*)$.

A d -rooted group is a finitely generated subgroup G of $\text{Aut } \Sigma^*$. The *rooted automorphism* is the automorphism $a \in \Sigma^*$ defined by

$$a(\sigma_1\sigma_2 \dots \sigma_n) = (\sigma_1 + 1)\sigma_2 \dots \sigma_n.$$

Fix a rooted group G , let $\text{Stab}_G(\sigma)$ be the stabilizer in G of the vertex $\sigma \in \Sigma^*$, and set $\text{Stab}_G(n) = \bigcap_{\sigma \in \Sigma^n} \text{Stab}_G(\sigma)$. Restriction induces a map

$$\pi_\sigma : \text{Stab}_G(\sigma) \rightarrow \text{Aut}(\sigma\Sigma^*) \rightarrow \text{Aut } \Sigma^*.$$

The group G is *fractal* if $\pi_\sigma \text{Stab}_G(\sigma) = G$ for all $\sigma \in \Sigma^*$, and *semi-fractal* if $\pi_\sigma \text{Stab}_G(\sigma) \leq G$ for all $\sigma \in \Sigma^*$. In that case, the map

$$\psi = (\psi_1, \dots, \psi_d)\omega : \text{Stab}_G(1) \rightarrow G^\Sigma$$

defined by $\psi_i(g) = \pi_i(g|_{\Sigma^*})$ is an embedding. It extends to a map still written $\psi : G \rightarrow G \wr \mathfrak{S}_\Sigma$, by lifting ψ to G using the natural map $G \rightarrow \mathfrak{S}_\Sigma$ given by restriction to the first level of the tree.

The *rigid stabilizer* of the vertex σ is

$$\text{Rist}_G(\sigma) = \bigcap_{\tau \notin \sigma\Sigma^*} \text{Stab}_G(\tau)$$

and the *rigid level stabilizer* of level n is

$$\text{Rist}_G(n) = \prod_{\sigma \in \Sigma^n} \text{Rist}_G(\sigma).$$

Note $\text{Rist}_G(\sigma) < \text{Stab}_G(\sigma)$ and $\text{Rist}_G(n) < G$ for all $\sigma \in \Sigma^*$ and $n \in \mathbb{N}$.

The group G is *level-transitive* if it acts transitively on Σ^n for all $n \in \mathbb{N}$. In that case, $\text{Stab}_G(\sigma)$ and $\text{Rist}_G(\sigma)$ depend, up to isomorphism, only on the length of σ .

Definition 3.3. The group G is a *branch* group if it is level-transitive, and $\text{Rist}_G(n)$ has finite index in G for all n . It is *weak branch* if all $\text{Rist}_G(\sigma)$ are non-trivial (and hence infinite). It is *regular branch* if $[G : \pi_\sigma \text{Rist}_G(\sigma)]$ is (finite and) constant for all long enough $\sigma \in \Sigma^*$. In that case, there is a finite-index subgroup $K \leq G$ such that $K^\Sigma \leq \psi(K)$, and G is *regular branch over* K .

Definition 3.4. Let G be a branch group generated by a finite set S , and consider the induced word metric on G . We say G is *contracting* if there is a constant D such that for every word $w \in S^*$ representing an element of $\text{Stab}_G(1)$, writing $\psi(w) = (w_1, \dots, w_d)$, we have

$$(5) \quad |w_i| < |w| \text{ for all } i \in \Sigma, \text{ as soon as } |w| > D.$$

It then follows that there is an algorithm \mathcal{A} solving the word problem in G : in this algorithm, we only assume that given a group generator we know its action on the top level Σ of the tree, and that given a word representing an element of $\text{Stab}_G(1)$ we may compute explicitly $\psi(w)$.

Initialization: Let $V \subset S^*$ be the set of all words of length at most D , and let $W \subset V$ be the set of words acting trivially on Σ . Note that ψ is a well-defined map from W to V^d . Assign to each $v \in V$ a flag, that is either “trivial”, “non-trivial” or “unknown yet”. Initially all flags are “unknown yet”.

For each $v \in V$ flagged “unknown yet”, if $v \in W$ or $\psi(v)$ has a component flagged “non-trivial”, flag v as “non-trivial”. If $\psi(v)$ has all components flagged “trivial”, flag v as “trivial”. Repeat the above procedure until no more flags are changed. Then flag all “unknown yet” words as “trivial”.

Computation: Let $w \in S^*$ be a word of which one asks whether it is trivial in G . If w belongs to V , its flag answers the word problem. If w acts non-trivially on Σ , it is non-trivial. Finally, if w acts trivially on Σ , write $\psi(w) = (w_1, \dots, w_n)$. By property (5), each w_i is strictly shorter than w , so the algorithm can be applied inductively to it. w is trivial if and only if all w_i are trivial.

Only one point deserves a special justification, and that is the flagging of “unknown yet” words as trivial. This is because such words act trivially on the tree, so belong to $\bigcap_{n \geq 0} \text{Stab}_G(n)$, which by assumption is trivial.

3.1. Proof of Theorem 3.1.

Lemma 3.5. Let $G = \langle S \rangle$ be a finitely generated group and let $H = \langle T \rangle$ be a finite-index subgroup of G , for some $T \subset S^*$. Let \tilde{S} be a set in bijection with S , and for $w = s_1 \dots s_n \in S^*$ set $\tilde{w} = \tilde{s}_1 \dots \tilde{s}_n \in \tilde{S}^*$.

There exists a finitely presented group $\Gamma = \langle \tilde{S} \rangle$ such that $\pi : \Gamma \xrightarrow{\tilde{s} \mapsto s} G$ is an epimorphism, and $\pi^{-1}(H) = \langle \tilde{T} \rangle$ in Γ .

Proof. Consider first F_S with its natural projection $\pi : F_S \rightarrow G$, and set $\Delta = \pi^{-1}(H)$. Since Δ has finite index in F_S , it is finitely generated, say by the set U . Our purpose is to find a quotient of F_S in which Δ is generated by T . For each $u \in U$, let w_u be an expression of $\pi(u)$ over T . It then suffices to consider

$$\Gamma = \langle S \mid u^{-1}w_u \forall u \in U \rangle.$$

□

In words, Γ a finitely presented group such that the subgroup lattice between G and $\langle T \rangle$ is isomorphic to the lattice between Γ and $\langle \tilde{T} \rangle$, where the different $\langle T \rangle$ ’s lie in different groups.

Proof of Theorem 3.1. Let G be regular branch on its subgroup K , and fix generating sets S for G and T for K . It loses no generality to assume $K \leq \text{Stab}_G(1)$, since one may always replace K by $K \cap \text{Stab}_G(1)$. Let Γ_0 be the group given by Lemma 3.5 for $H = K$. Let Δ_0 be the natural lift of $\text{Stab}_G(1)$ to Γ_0 ; and let Υ_0 be the natural lift of K to Γ_0 .

Let U be a generating set of $\text{Stab}_1(G)$ (so Δ_0 is generated by \tilde{U}), and let $\tilde{\psi} : \Delta_0 \rightarrow \Gamma_0^d$ be the natural lift of $\psi : \text{Stab}_G(1) \rightarrow G^d$; it maps \tilde{u} to $\tilde{\psi(u)}$, where the wide tilde is applied to all d factors of $\psi(u)$. Note that $\tilde{\psi}$ satisfies the contracting condition for the same constant D as ψ .

Since G is regular branch, there is an embedding $1^i \times K \times 1^{d-1-i} \hookrightarrow K$, from which for each generator $t \in T$ of K we may choose a word $\phi_i(t) \in T^*$ such that $\psi(\phi_i(t)) = (1, \dots, t, \dots, 1)$ with the ‘ t ’ in position i .

Now write $\tilde{\psi}(\phi_i(t)) = (r_{t,1}, \dots, r_{t,i}t, \dots, r_{t,d})$ for some $r_{t,i} \in \Upsilon_0$. These elements’ images in K are trivial, since $\tilde{\psi}$ is a lift of ψ . Furthermore, since $\tilde{\psi}$ is contracting, one may replace $\{r_{t,i}\}_{t \in T, i \in \Sigma}$ by its iterates under all $\pi_i \tilde{\psi}$, where π_i is the projection on the i -th factor, and still obtain a finite set of relations.

Let Γ be the quotient of Γ_0 by this sets’ normal closure. Then Γ is finitely presented and surjects onto G (since $r_{t,i} \cong 1$ in G). Let Δ and Υ be the images of Δ_0 and Υ_0 in Γ , and note that ψ lifts again to $\tilde{\psi}$ on Γ , because the new relators $r_{t,i}$ map to other new relators.

The data are summed up in the following diagram, which should be viewed as a “chair with ψ and $\tilde{\psi}$ coming forward”:

$$\begin{array}{ccccccc}
 & & \Gamma & \xrightarrow{\quad} & G & & \\
 & & \downarrow & & \downarrow & & \\
 \Gamma^d & \xleftarrow{\tilde{\psi}} & \Delta & \xrightarrow{\quad} & \text{Stab}_G(1) & \xrightarrow{\psi} & G^d \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Upsilon^d & & \Upsilon & \xrightarrow{\quad} & K & \xleftarrow{\quad} & K^d
 \end{array}$$

Since $\text{Im } \tilde{\psi}$ contains Υ^d , it has finite index in Γ^d . Since Γ^d is finitely presented, $\text{Im } \tilde{\psi}$ too is finitely presented. Similarly, Δ is finitely presented, and we may express $\ker \tilde{\psi}$ as the normal closure $\langle R_1 \rangle^\#$ in Δ of those relators in $\text{Im } \tilde{\psi}$ that are not relators in Δ . Clearly R_1 may be chosen to be finite.

We now use the assumption that G is contracting, with constant D . Let R_2 be the set of words over S of length at most D that represent the identity in G . Set $R = R_1 \cup R_2$, which clearly is finite.

We now consider T as a set distinct from S , and not as a subset of S^* . We extend each ϕ_i to a monoid homomorphism $(S \cup T)^* \rightarrow (S \cup T)^*$ by defining it arbitrarily on S .

Assume $\Gamma = \langle S | Q \rangle$, and let $w_t \in S^*$ be a representation of $t \in T$ as a word in S . We claim that the following is an L -presentation of G :

$$(6) \quad G = \langle S \cup T | Q \cup \{t^{-1}w_t\}_{t \in T} | \{\phi_i\}_{i \in \Sigma} | R_1 \cup R_2 \rangle.$$

For this purpose, consider the following subgroups Ξ_n of Γ : first $\Xi_0 = \{1\}$, and by induction

$$\Xi_{n+1} = \{\gamma \in \Delta | \tilde{\psi}(\gamma) \in \Xi_n^d\}.$$

We computed $\Xi_1 = \langle R \rangle^\#$. Since G acts transitively on the n -th level of the tree, a set of normal generators for Ξ_n is given by $\bigcup_{i \leq n} \phi^i(R)$, where ϕ is any choice of ϕ_i for $i \in \Sigma$. We also note that $\psi(\Xi_{n+1}) = \Xi_n^d$.

We will have proven the claim if we show $G = \Gamma / \bigcup_{n \geq 0} \Xi_n$. Let then $w \in \Gamma$ represent the identity in G . Applying to it $|w|$ times the map ψ , we obtain $d^{|w|}$ words that are all of length at

most K , that is, that belong to Ξ_1 . Then since $\psi(\Xi_{n+1}) = \Xi_n^d$, we get $w \in \Xi_{|w|+1}$, and (6) is a presentation of G .

As a bonus, the presentation (6) expresses K as the subgroup of G generated by T . \square

A few remarks are in order. First, one can usually do with only one substitution, say ϕ_1 , since in many cases the other ϕ_i are conjugates of ϕ_1 . Second, ϕ_1 induces an isomorphism from K to its subgroup $K \times 1^{d-1}$, so there is a one-step HNN extension of G that is finitely presented — namely, the extension identifying K and $K \times 1^{d-1}$. Third, in many cases (but not all) ϕ_1 can be extended to an endomorphism of G ; in that case, one may delete T from the generating set and obtain an ascending L -presentation.

In all cases, K admits an ascending L -presentation, so embeds in a finitely presented group L , and $\langle G, L \rangle$ is a finite extension of L , hence is a finitely presented group containing G .

Proof of Theorem 3.2. Since G is contracting, there is a constant D such that $|w_i| < |w|$ whenever $|w| > D$. This implies, using the triangular inequality, that there are constants $\eta < 1$ and D' such that $|w_i| < \eta|w|$ whenever $|w| > D'$.

Now levels can be “collapsed” in a branch group: for any k we may consider the (same) action of G on $(\Sigma^k)^*$, with map ψ given by k -fold composition of the original map ψ . The resulting group action is still branch.

However, the result of this process is that the constant η above can be replaced by any power of itself, say $\frac{1}{2}$, at the cost of enlarging the branching number of the tree.

The generating set then now be replaced by a ball of sufficiently large radius, so that the constant L becomes 1.

We have reached a “canonical situation”, where the maps ψ and $\tilde{\psi}$ satisfy $|w_i| \leq \frac{1}{2}(|w| + 1)$ for all w .

Assume now by contradiction that G is finitely presented, say $G = \langle S | R \rangle$ with $\pi : F_S \rightarrow G$ the canonical map, and assume that the greatest length among the relators is minimal. All $r \in R$ being trivial in G , satisfy *a fortiori* $\pi(r) \in \text{Stab}_G(1)$, so $\tilde{\psi}(r) = (r_1, \dots, r_d)$ is well defined. By the Reidemeister-Schreier process, a presentation of $G \times 1 \times \dots \times 1$ is

$$\langle S | r_i \text{ for all } r \in R \text{ and } i \in \Sigma \rangle.$$

By our assumptions that $|r_i| \leq \frac{1}{2}(|r| + 1)$ and $\max |r|$ is minimal, we must have $|r| \leq 1$ for all relations, so G is free. However a free group may not contain commuting subgroups with trivial intersection, like $K \times 1 \times \dots \times 1$ and $1 \times \dots \times 1 \times K$. This is our required contradiction. \square

3.2. Schur multipliers. In his paper [Gri99] Rostislav Grigorchuk computed explicitly the Schur multiplier $H_2(G, \mathbb{Z})$ of his group — he proved that $H_2(G) = (\mathbb{Z}/2)^\infty$. We outline here a general computation $H_2(G, \mathbb{Z})$ for branch groups G .

Theorem 3.6. *Let G be a finitely generated, contracting, semi-fractal, regular branch group. Then $H_2(G, \mathbb{Z}) \cong A \oplus B^\infty$, for finite abelian groups A, B .*

As a consequence, all such groups are infinitely presented.

Proof. We concentrate on the exact sequence $1 \rightarrow K^d \rightarrow K \rightarrow Q \rightarrow 1$, for some finite group Q . By (4) and (3), $H_2(K, \mathbb{Z}Q) = H_2(K^d, \mathbb{Z}) = H_2(K, \mathbb{Z})^d \oplus (K/K' \otimes K/K')^{d(d-1)/2}$. Taking Q -invariants of the right-hand side collapses all d copies of $H_2(K, \mathbb{Z})$ together, but we are left with the equation $H_2(K, \mathbb{Z}) = H_2(K, \mathbb{Z}) \oplus B$, where B , a quotient of $(K/K' \otimes K/K')^{d(d-1)/2}$, is a finite group.

Then $H_2(G, \mathbb{Z})$ is obtained from $H_2(K, \mathbb{Z})$ by extension and quotient by finite-rank abelian groups, and the claimed result follows. \square

Note, as a corollary, that if K is perfect, then it is a finitely presented, infinitely related group with trivial Schur multiplier.

3.3. Perfect branch groups. We consider a class of branch groups, of special interest for being perfect. They form a subclass of the GGS groups studied in [BŠ00]. Let A be a finite, perfect, group acting transitively on Σ , with two elements $* \neq \dagger \in \Sigma$ such that $\text{Stab}_A(*) \setminus \text{Stab}_A(\dagger) \neq \emptyset$ (think for instance \mathfrak{A}_5).

Let \bar{A} be a copy of A , and consider $\Gamma = A * \bar{A}$. Define an action of Γ on Σ^* by

$$\begin{aligned} (\sigma_1 \sigma_2 \dots \sigma_n)^a &= \sigma_1^a \sigma_2 \dots \sigma_n, \\ (\sigma_1 \sigma_2 \dots \sigma_n)^{\bar{a}} &= \begin{cases} \sigma_1 \sigma_2^a \sigma_3 \dots \sigma_n & \text{if } \sigma_1 = *, \\ \sigma_1 (\sigma_2 \sigma_3 \dots \sigma_n)^{\bar{a}} & \text{if } \sigma_1 = \dagger, \\ \sigma_1 \sigma_2 \dots \sigma_n & \text{otherwise.} \end{cases} \end{aligned}$$

Let G be the group defined by this action, i.e. the quotient of Γ by the kernel of the action.

Proposition 3.7. *G is a perfect finitely generated regular branch group over itself.*

Proof. Clearly G is perfect, being generated by two perfect groups, and finitely generated, being generated by two finite groups.

Note now that $\text{Stab}_G(1) = \bar{A}^G$. The map $\psi : G \rightarrow G \wr_\Sigma A$ is given by

$$\psi(a) = (1, \dots, 1)a, \quad \psi(\bar{a}) = (a, 1, \dots, 1, \bar{a})1,$$

where in this last expression the ‘ a ’ is at position $*$ and the ‘ \bar{a} ’ is at position \dagger . The conditions on A imply that it contains an element x moving \dagger but not $*$. The computation $\psi[\bar{a}, \bar{b}^x] = ([a, b], 1, \dots, 1)$ shows that $\psi(G)$ contains $A \times 1 \cdots \times 1$, since A is perfect; then $\psi(G)$ contains too

$$(a, 1, \dots, 1)^{-1} \psi(\bar{a}) = (a^{-1}, 1, \dots, 1)(a, 1, \dots, 1, \bar{a}) = (1, \dots, 1, \bar{a}),$$

so $\psi(G)$ contains $1 \times \cdots \times 1 \times \bar{A}$, and since A is Y -transitive it contains $G \times \cdots \times G$. (Explicitly, we have $\psi^{-1}(G^\Sigma) = \text{Stab}_G(1)$.) \square

In this context, the statements of the previous section simplify: we have perfect regular branch groups G with $H_2(G, \mathbb{Z}) = 0$, that are finitely L -presented but infinitely presented. The group $\Gamma = A * \bar{A}$ is the same as the Γ of Theorem 3.1, and the subgroup Δ is $*_{a \in A} \bar{A}^a$. The map $\tilde{\psi}$ is given by

$$\tilde{\psi}(\bar{a}^b) = (a \text{ in position } *, \bar{a} \text{ in position } \dagger^b).$$

Let ϕ be some word substitution on Γ mapping g to $(g, 1, \dots, 1)$, as given by the computations in the previous theorem. We then have an L -presentation

$$G = \left\langle A, \bar{A} \left\| \phi \left| \begin{array}{ll} \bar{a}^{1-b} & \text{whenever } *^b = *, \dagger^b = \dagger \\ \bar{a}^{1-b+c-d} & \text{whenever } *^b = *, \dagger^b = \dagger^c, *^c = *^d, \dagger^d = \dagger \text{ are all distinct} \\ \bar{a}^{1-b+c-d(1-b+c)} & \text{whenever } *^b = \dagger^c = \dagger^d = *, \dagger^b = *^c, *^d = \dagger \text{ are all distinct} \\ [\phi(a), \phi(b)^c] & \text{whenever } *^c \neq * \end{array} \right. \right. \right\rangle.$$

Indeed, the first three relations identify all products $\bar{a}^{b_1 + \cdots + b_n}$ with same ψ -image, and the last ones are the commutation relations lifted from $\Gamma \times \cdots \times \Gamma$.

4. EXAMPLES

We start by an example of wreath product:

Theorem 4.1. *The following is an L -presentation of the “lamplighter group” $G = \mathbb{Z}/2 \wr \mathbb{Z}$:*

$$G = \langle a, b, t \mid a^2, a^{-1}b \mid \phi \mid [a, b] \rangle,$$

where $\phi : \{a, b, t\}^* \rightarrow \{a, b, t\}^*$ is given by

$$\phi(a) = a, \quad \phi(b) = b^t, \quad \phi(t) = t.$$

However, this group admits no finite presentation.

Proof. A presentation of G is

$$\langle a, t \mid a^2, [a, a^{t^i}] \forall i \in \mathbb{Z} \rangle.$$

By conjugating the last relation by t^i , we may assume the set of relators is a^2 and the $[a, a^{t^i}]$ with $i \geq 0$. The latter are precisely the relators obtained from $\phi^i([a, b])$ by applying the relation $a = b$.

It follows from [Bau61] that G is not finitely presented. Even better, (2) gives $H_2(\mathbb{Z}/2 \wr \mathbb{Z}, \mathbb{Z}) = (\mathbb{Z}/2)^\infty$. \square

Note however the following seemingly similar example, due to Gilbert Baumslag, which is finitely L -presented by arguments similar to those in Theorem 4.1:

Theorem 4.2 ([Bau71]). *The group*

$$G = \langle a, b, t \mid a^{t-4}, b^{2t-1}, [a, b^{t^i}] \forall i \in \mathbb{Z} \rangle$$

is an infinitely-presented, metabelian group, with $H_2(G, \mathbb{Z}) = 0$.

This example was devised to show that the Schur multiplier’s rank may be much smaller than the number of relators. In that view, we may ask the following question:

Problem 4.3. *Do there exist non-finitely- L -presented groups with trivial Schur multiplier?*

An interesting example of group acting on a rooted tree is the “Brunner-Sidki-Vieira group”; we rephrase their result in terms of L -presentations:

Theorem 4.4 ([BSV99], Proposition 15). *Consider the group $G = \langle \mu, \tau \rangle$ acting on $\{1, 2\}^*$, with $\psi(a^{-1}\mu) = (1, \mu^{-1})$ and $\psi(a^{-1}\tau) = (1, \tau)$ (so τ and μ act like a on the top node of the tree. Note that G is neither rooted nor branch, though it is “weakly branch” [Gri00].) Writing $\lambda = \tau\mu^{-1}$, G admits the ascending L -presentation*

$$G = \langle \lambda, \tau \mid \phi \mid [\lambda, \lambda^\tau], [\lambda, \lambda^{\tau^3}] \rangle,$$

where ϕ is defined by $\tau \mapsto \tau^2$ and $\lambda \mapsto \tau^2\lambda^{-1}\tau^2$.

We may even conclude that $H_2(G, \mathbb{Z}) = (\mathbb{Z} \times \mathbb{Z})^\infty$, freely generated by the images of $\phi^n[\lambda, \lambda^\tau]$ and $\phi^n[\lambda, \lambda^{\tau^3}]$.

We now give presentations for four of the five “testbed groups” studied in [BG00, BG99].

4.1. An L -presentation for G . The group G , the *first Grigorchuk group*, is the 2-rooted group $G = \langle a, b, c, d \rangle$, with a the rooted element and b, c, d defined by

$$\psi(b) = (a, c), \quad \psi(c) = (a, d), \quad \psi(d) = (1, b).$$

G is a regular branch group over $K = \langle (ab)^2 \rangle^\#$.

Theorem 4.5. *The Grigorchuk group G admits the ascending L -presentation*

$$G = \langle a, c, d \mid \sigma \mid a^2, [d, d^a], [d^{ac}, (d^{ac})^a] \rangle,$$

where $\sigma : \{a, c, d\}^* \rightarrow \{a, c, d\}^*$ is defined by

$$\sigma(a) = aca, \quad \sigma(c) = cd, \quad \sigma(d) = c.$$

Proof. Rephrasing of [Lys85]. \square

4.2. An L -presentation for \tilde{G} . The group \tilde{G} , the *Grigorchuk supergroup*, is the 2-rooted group $G = \langle a, \tilde{b}, \tilde{c}, \tilde{d} \rangle$, with a the rooted element and $\tilde{b}, \tilde{c}, \tilde{d}$ defined by

$$\psi(\tilde{b}) = (a, \tilde{c}), \quad \psi(\tilde{c}) = (1, \tilde{d}), \quad \psi(\tilde{d}) = (1, \tilde{b}).$$

\tilde{G} is a regular branch group over $\tilde{K} = \langle (a\tilde{b})^2, (a\tilde{d})^2 \rangle^\#$. It is named thus because it contains G as a subgroup.

Theorem 4.6. *The group \tilde{G} admits the ascending L -presentation*

$$\tilde{G} = \langle a, \tilde{b}, \tilde{c}, \tilde{d} \mid \tilde{\sigma} \mid a^2, [\tilde{b}, \tilde{c}], [\tilde{c}, \tilde{c}^a], [\tilde{c}, \tilde{d}^a], [\tilde{d}, \tilde{d}^a], [\tilde{c}^{a\tilde{b}}, (\tilde{c}^{a\tilde{b}})^a], [\tilde{c}^{a\tilde{b}}, (\tilde{d}^{a\tilde{b}})^a], [\tilde{d}^{a\tilde{b}}, (\tilde{d}^{a\tilde{b}})^a] \rangle,$$

where $\tilde{\sigma} : \{a, \tilde{b}, \tilde{c}, \tilde{d}\}^* \rightarrow \{a, \tilde{b}, \tilde{c}, \tilde{d}\}^*$ is defined by

$$a \mapsto a\tilde{b}a, \quad \tilde{b} \mapsto \tilde{d}, \quad \tilde{c} \mapsto \tilde{b}, \quad \tilde{d} \mapsto \tilde{c}.$$

Proof. Rephrasing of [BG99, Proposition 5.6]. □

4.3. An L -presentation for Γ . The group Γ , the *Fabrykowski-Gupta group*, is the 3-rooted group $G = \langle a, r \rangle$, with a the rooted element and r defined by

$$\psi(r) = (a, 1, r).$$

Γ is a regular branch group over $\Gamma' = \langle [a, r] \rangle^\#$.

Theorem 4.7. *The Fabrykowski-Gupta group Γ admits the ascending L -presentation*

$$\langle a, r \mid \sigma, \chi_1, \chi_2 \mid a^3, [r^{1+a^{-1}-1+a+1}, a], [a^{-1}, r^{1+a+a^{-1}}][r^{a+1+a^{-1}}, a] \rangle,$$

where $\sigma, \chi_1, \chi_2 : \{a, r\}^* \rightarrow \{a, r\}^*$ are given by

$$\begin{aligned} \sigma(a) &= r^{a^{-1}}, & \sigma(r) &= r, \\ \chi_1(a) &= a, & \chi_1(r) &= r^{-1}, \\ \chi_2(a) &= a^{-1}, & \chi_2(r) &= r. \end{aligned}$$

Proof. We follow Theorem 3.1. Consider first the group $F = \langle a, r \mid a^3, r^3 \rangle$. Clearly, $F/F' \cong (\mathbb{Z}/3)^2 \cong \Gamma/\Gamma'$. Using the computer algebra program GAP, we compute a presentation for $\text{Im } \tilde{\psi}$, and rewrite its relators as words in X , where X is a generating set for Γ' . We also construct a group homomorphism $\sigma_0 : \Gamma' \rightarrow 1 \times 1 \times \Gamma'$. Then Theorem 3.1 gives a finite L -presentation for Γ with generators $\{a, r\} \cup X$.

We now note that σ_0 can be extended to a homomorphism $\sigma : \Gamma \rightarrow A \times R \times \Gamma$, where $A = \langle a \rangle$ and $R = \langle r \rangle$ have order 3. The substitution σ can be used instead of σ_0 , giving rise to a simpler presentation with generators a, r .

Finally, we note that the presentation can be simplified from 6 iterated relators to 2 by introducing two extra substitutions χ_1, χ_2 induced by group automorphisms. □

Note that for G and \tilde{G} the iterated relations are of the form $[x, x^a]$ where x belongs to a first-level rigid stabilizer. For Γ , however, one obtains fewer relations by considering more general expressions, as above.

4.4. An L -presentation for $\bar{\Gamma}$. The group $\bar{\Gamma}$, introduced in [BG00], is the 3-rooted group $G = \langle a, s \rangle$, with a the rooted element and s defined by

$$\psi(s) = (a, a, s).$$

Set $x = ta^{-1}$, $y = a^{-1}t$ and $K = \langle x, y \rangle$, a torsion-free index-3 subgroup of $\bar{\Gamma}$.

Theorem 4.8. *The groups K and $\bar{\Gamma}$ are not branch, but are finitely L -presented.*

Proof. We start by computing an L -presentation for K . As above, $\psi(K')$ contains $K' \times K' \times K'$; but $K/K' \cong \mathbb{Z}^2$ and neither $\bar{\Gamma}$ nor K are branch.

First we chose generators of $\text{Stab}_K(1)$:

$$\begin{aligned}\alpha &= x^{-1}y = (x, 1, x^{-1}), & \beta &= y^{-1}x^{-1}y^{-1} = (y, 1, y^{-1}), \\ \gamma &= x^{-1}y^{-1}x^{-1} = (1, x, x^{-1}), & \delta &= xy^{-1} = (1, y, y^{-1}).\end{aligned}$$

Then, we chose generators of K' :

$$\begin{aligned}e &= \beta^{-1}\delta\gamma = [y, xyx] = (y^{-1}, yx, x^{-1}), & f &= \gamma\beta^{-1}\delta = [x^{-1}y^{-1}x^{-1}, y] = (y^{-1}, xy, x^{-1}), \\ g &= \gamma^{-1}\alpha\beta = [x, y] = (xy, x^{-1}, y^{-1}), & h &= \beta\gamma^{-1}\alpha = [y^{-1}x^{-1}, y^{-1}] = (yx, x^{-1}, y^{-1}).\end{aligned}$$

The fact that K' is normal can be seen in the following conjugation relations:

\swarrow	e	f	g	h
x	$g^{-1}hf^{-1}g^{-1}$	$f^{-1}g^{-1}$	e	$g^{-1}he$
x^{-1}	ehf^{-1}	$h^{-1}e^{-1}$	$h^{-1}f^{-1}$	
y	$g^{-1}e^{-1}$	$h^{-1}e^{-1}$	$g^{-1}eh$	f
y^{-1}	$e^{-1}gf$	h	$f^{-1}g^{-1}$	$f^{-1}g^{-1}ef$

Define now the group L by its generators $S = \{x^{\pm 1}, y^{\pm 1}\}$ and relators $k^s = w_{k,s}$ for $s \in S$ and $k \in \{e, f, g, h\}$, where $w_{k,s}$ is the word in the above table. Note then that L' is generated by the words e, f, g, h .

As in the proof of Theorem 3.1, consider the map $\tilde{\psi} : \text{Stab}_L(1) \rightarrow L^3$ corresponding to $\psi : \text{Stab}_K(1) \rightarrow K^3$, and by the Reidemeister-Schreier method compute a presentation for $\tilde{\psi}(\text{Stab}_L(1))$.

Assume $L = \langle S | Q \rangle$. A presentation for L^3 is $\langle S_1 \cup S_2 \cup S_3 | Q_1 \cup Q_2 \cup Q_3 \cup [S_i, S_{j \neq i}] \rangle$. The image of $\tilde{\psi}$ can be described as $\{(u, v, w) \in L^3 | uvw \in L'\}$. We choose $\{x_3^m y_3^n\}_{m,n \in \mathbb{Z}}$ as Schreier transversal for this subgroup, and denote the Schreier generators

$$u_{imn} = x_3^m y_3^n x_i y_3^{-n} x_3^{-m-1}, \quad v_{imn} = x_3^m y_3^n y_i y_3^{-n-1} x_3^{-m}.$$

Then these generators satisfy

$$\begin{aligned}u_{1mn} &= u_{100} = \alpha, & v_{1mn} &= u_{200}^{-m} v_{100} u_{200}^m = \gamma^{-m} \beta \gamma^m, \\ u_{2mn} &= u_{200} = \gamma, & v_{2mn} &= u_{100}^{-m} v_{200} u_{100}^m = \alpha^{-m} \delta \alpha^m, \\ u_{3mn} &= u_{200}^{-m} v_{100}^{-n} u_{200}^{-1} v_{100}^n u_{200}^{m+1}, & v_{3mn} &= 1.\end{aligned}$$

The relators we obtain, in terms of $\alpha, \beta, \gamma, \delta$, are

$$\begin{aligned}& [\alpha, \gamma], [\alpha\beta, \gamma\delta], \\ & [\alpha\gamma^{-1}, \beta^{-n}\gamma^{-1}\beta^n], [\gamma^{-n}\beta\gamma^n, \alpha^{-n}\delta\alpha^n] \text{ for all } n \in \mathbb{Z}, \\ & w(\alpha\gamma^{-1}, \beta), w(\alpha^{-1}\gamma, \delta), w(\alpha^{-1}, \delta^{-1}) \text{ for all } w \in Q.\end{aligned}$$

This clearly gives a finite L -presentation for $\tilde{\psi}(\text{Stab}_L(1))$ — compare with the proof of Theorem 4.1. Now the computation of a presentation for K can be finished as in the proof of Theorem 3.1.

Finally, a finite L -presentation for G can be obtained using Proposition 2.8. \square

4.5. An L -presentation for $\bar{\bar{\Gamma}}$. The group $\bar{\bar{\Gamma}}$, the *Gupta-Sidki group*, is the 3-rooted group $G = \langle a, t \rangle$, with a the rooted element and t defined by

$$\psi(t) = (a, a^{-1}, t).$$

$\bar{\bar{\Gamma}}$ is a regular branch group over $\bar{\bar{\Gamma}}' = \langle [a, t] \rangle^\#$.

Theorem 4.9. *The Gupta-Sidki group $\bar{\Gamma}$ admits the L -presentation*

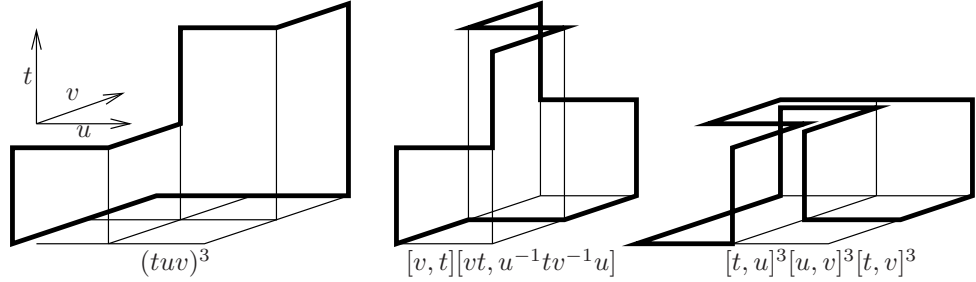
$$\langle a, t, u, v \mid a^3, t^3, u^{-1}t^a, v^{-1}t^{a^{-1}} \mid \sigma, \chi \mid (tuv)^3, [v, t][vt, u^{-1}tv^{-1}u], [t, u]^3[u, v]^3[t, v]^3 \rangle,$$

where $\sigma, \chi : \{t, u, v\}^* \rightarrow \{t, u, v\}^*$ are given by

$$\sigma : \begin{cases} t \mapsto t, \\ u \mapsto [u^{-1}t^{-1}, t^{-1}v^{-1}]t = u^{-1}tv^{-1}tuvt^{-1}, \\ v \mapsto t[tv, ut] = t^{-1}vutv^{-1}tu^{-1}, \end{cases} \quad \chi : \begin{cases} t \mapsto t^{-1}, \\ u \mapsto u^{-1}, \\ v \mapsto v^{-1}. \end{cases}$$

Note that χ is induced by the automorphism of $\bar{\Gamma}$ defined by $a \mapsto a$, $t \mapsto t^{-1}$; however, σ does not extend to an endomorphism of $\bar{\Gamma}$.

Note also that all the iterated relators can be expressed as words over $\{t, u, v\}$ with 0-sum in each variable. Their most natural representation is as closed paths in the $\{t, u, v\}$ -space:



Then the fact that these elements are non-trivial relations translates to the fact that their projection on any plane $t = -u$, $u = -v$ or $v = -t$ gives a trivial path (up to $t^3 = u^3 = v^3 = 1$), while they themselves are not trivial paths. Incidentally, these projections are none but the $\psi_i : \langle t, u, v \rangle \rightarrow \langle a, t \rangle$, for $i \in \Sigma$.

Proof. We follow Theorem 3.1. Consider first the group $F = \langle a, t \mid a^3, t^3 \rangle$. Clearly, $F/F' \cong (\mathbb{Z}/3)^2 \cong \bar{\Gamma}/\bar{\Gamma}'$. Using the computer algebra program GAP, we compute a presentation for $\text{Im } \tilde{\psi}$, and rewrite its relators as words in X , where X is a generating set for $\bar{\Gamma}'$. We also construct a group homomorphism $\sigma_0 : \bar{\Gamma}' \rightarrow 1 \times 1 \times \bar{\Gamma}'$. Then Theorem 3.1 gives a finite L -presentation for $\bar{\Gamma}$ with generators $\{a, t\} \cup X$.

We now note that σ_0 can be extended to a homomorphism $\sigma : \text{Stab}_{\bar{\Gamma}}(1) \rightarrow A \times A \times \bar{\Gamma}$, where $A = \langle a \rangle$ has order 3. The substitution σ can be used instead of σ_0 , giving rise to a simpler presentation with generators a, t, u, v , where $t, u = t^a, v = t^{a^{-1}}$ is a generating set for $\text{Stab}_{\bar{\Gamma}}(1)$.

Finally, we note that the presentation can be simplified from 6 iterated relators to 3 by introducing an extra substitution χ , induced by a group automorphism. \square

Problem 4.10. *Does there exist a finite ascending L -presentation for $\bar{\Gamma}$?*

In these examples, easy computations yield $H_2(G, \mathbb{Z}) = H_2(\tilde{G}, \mathbb{Z}) = (\mathbb{Z}/2)^\infty$ and $H_2(\Gamma, \mathbb{Z}) = H_2(\bar{\Gamma}, \mathbb{Z}) = (\mathbb{Z}/3)^\infty$.

REFERENCES

- [Aan73] Stål Aanderaa, *A proof of Higman's embedding theorem using Britton extensions of groups*, Word problems: decision problems and the Burnside problem in group theory (Conf. on Decision Problems in Group Theory, Univ. California, Irvine, Calif., 1969; dedicated to Hanna Neumann), North-Holland, Amsterdam, 1973, pp. 1–18. Studies in Logic and the Foundations of Math., Vol. 71.
- [AC80] Stål Aanderaa and Daniel E. Cohen, *Modular machines and the Higman-Clapham-Valiev embedding theorem*, Word problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976), North-Holland, Amsterdam, 1980, pp. 17–28.

- [Bau61] Gilbert Baumslag, *Wreath products and finitely presented groups*, Math. Z. **75** (1960/1961), 22–28.
- [Bau71] Gilbert Baumslag, *A finitely generated, infinitely related group with trivial multiplier*, Bull. Austral. Math. Soc. **5** (1971), 131–136.
- [BG99] Laurent Bartholdi and Rostislav I. Grigorchuk, *On parabolic subgroups and Hecke algebras of some fractal groups*, submitted to Proc. Conf. Bielefeld, 1999, math.GR/9911206.
- [BG00] Laurent Bartholdi and Rostislav I. Grigorchuk, *Spectra of non-commutative dynamical systems and graphs related to fractal groups*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 6, 429–434.
- [Bla72] Norman Blackburn, *Some homology groups of wreath products*, Illinois J. Math. **16** (1972), 116–129.
- [Bro94] Kenneth S. Brown, *Cohomology of groups*, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original.
- [BŠ00] Laurent Bartholdi and Zoran Šuník, *On the word and period growth of some groups of tree automorphisms*, to appear in Comm. Algebra, 2000, math.GR/0005113.
- [BSV99] Andrew M. Brunner, Said N. Sidki, and Ana Cristina Vieira, *A just nonsolvable torsion-free group defined on the binary tree*, J. Algebra **211** (1999), no. 1, 99–114.
- [Bur97] William S. Burnside, *Note on the symmetric group*, Proc. London Math. Soc. **28** (1897), 119–129.
- [Day57] Mahlon M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.
- [Gri84] Rostislav I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985, English translation: Math. USSR-Izv. **25** (1985), no. 2, 259–300.
- [Gri98] Rostislav I. Grigorchuk, *An example of a finitely presented amenable group that does not belong to the class EG*, Mat. Sb. **189** (1998), no. 1, 79–100.
- [Gri99] Rostislav I. Grigorchuk, *On the system of defining relations and the Schur multiplier of periodic groups generated by finite automata*, Groups St. Andrews 1997 in Bath, I (N. Ruskuc C.M. Campbell, E.F. Robertson and G. C. Smith, eds.), Cambridge Univ. Press, Cambridge, 1999, pp. 290–317.
- [Gri00] Rostislav I. Grigorchuk, *Just infinite branch groups*, New horizons in pro- p groups (Markus P. F. du Sautoy Dan Segal and Aner Shalev, eds.), Birkhäuser Boston, Boston, MA, 2000, pp. 121–179.
- [Gup84] Narain D. Gupta, *Recursively presented two generated infinite p -groups*, Math. Z. **188** (1984), no. 1, 89–90.
- [Hal54] Philip Hall, *Finiteness conditions for soluble groups*, Proc. London Math. Soc. (3) **4** (1954), 419–436.
- [Hig61] Graham Higman, *Subgroups of finitely presented groups*, Proc. Roy. Soc. Ser. A **262** (1961), 455–475.
- [Kar87] Gregory Karpilovsky, *The Schur multiplier*, The Clarendon Press Oxford University Press, New York, 1987.
- [Lin73] Aristid Lindenmayer, *Cellular automata, formal languages and developmental systems*, Logic, methodology and philosophy of science, IV (Proc. Fourth Internat. Congr., Bucharest, 1971), North-Holland, Amsterdam, 1973, pp. 677–691. Studies in Logic and Foundations of Math., Vol. 74.
- [LS70] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Springer-Verlag, 1970.
- [Lys85] Igor G. Lysionok, *A system of defining relations for the Grigorchuk group*, Mat. Zametki **38** (1985), 503–511.
- [Moo97] Eliakim H. Moore, *Concerning the abstract groups of order $k!$ and $\frac{1}{2}k!$ holohedrally isomorphic with the symmetric and the alternating substitution groups on k letters*, Proc. London Math. Soc. **28** (1897), 357–366.
- [Neu67] Hanna Neumann, *Varieties of groups*, Springer-Verlag New York, Inc., New York, 1967.
- [Ol’70] Alexander Yu. Ol’shanskii, *The finite basis problem for identities in groups*, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 376–384.
- [OS] Alexander Yu. Ol’shanskii and Mark V. Sapir, *Length and area functions on groups and quasi-isometric Higman embeddings*, to appear in IJAC.
- [RS80] Grzegorz Rozenberg and Arto Salomaa, *The mathematical theory of L systems*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980.
- [S⁺93] Martin Schönert et al., *GAP: Groups, algorithms and programming*, RWTH Aachen, 1993.
- [Ser93] Vlad Sergiescu, *Graphes planaires et présentations des groupes de tresses*, Math. Z. **214** (1993), no. 3, 477–490.
- [Sid87] Said N. Sidki, *On a 2-generated infinite 3-group: the presentation problem*, J. Algebra **110** (1987), no. 1, 13–23.

E-mail address: laurent@math.berkeley.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
94720 BERKELEY
U. S. A.